Properties of irrotational vector fields

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Abstract. *In the first part we present properties of irrotational vector fields and in the second part properties of arbitrary vector fields, all arising from the study of the energy associated to the vector field. The results relate to critical points and extrema of the energy, the Riemannian structure on the manifold, the behaviour of the energy along orbits of the field and the type of these orbits.*

INTRODUCTION

Classical models which were used as starting point for our theory on irrotational vector fields meet in Gravitation theory, Electromagnetism (electrostatic fields and stationary magnetic fields), Hydrodynamics (the steady irrotational motion of inviscid fluids), Heat propagation theory in homogeneous, isotropic media, Steady magnetohydrodynamic flows (velocity fields, electrostatic fields) etc. Therefore our results give insight to some still oustanding questions in the above fields (for example the theory and design of magnetic gaps).

One of our theorems shows that if the speed of variation of the energy attached to a vector field does not vanish in any point, then the orbits of the field cannot be closed. Consequently, the flows generated by the vector fields which satisfy the hypothesis in this theorem do not present Hopf bifurcation.

1. Let (M, g) be a finite dimensional Riemannian manifold and $\mathscr{X}(M)$ the Lie algebra of C^∞ vector fields on M. Fix $X \in \mathscr{X}(M)$ and suppose that for all Y, $Z \in \mathscr{X}(M)$ we have

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Key-Words: Local potential, Riemannian manifold, energy, orbit.
1980 Mathematics Subject Classification: 53 C 21.

(1) $$g(\nabla_Y X, Z) - g(\nabla_Z X, Y) = 0$$

i.e. X is an *irrotational vector field* on (M, g).

The relation (1) is equivalent to the fact that $(\nabla X)_x$ is a symmetric endomorphism with respect to the scalar product induced by g in T_xM for each point $x \in M$. Hence the concurrent vector fields are irrotational.

A vector field is irrotational if and only if it is a locally potential field. An irrotational vector field X for which $\operatorname{div} X = 0$ is called *harmonic*.

Let $f = \frac{1}{2} g(X, X)$ be the *energy* of X. As

$$df(Y) = g(\nabla_Y X, X) = g(\nabla_X X, Y), \ \forall Y \in \mathscr{X}(M)$$

it follows

$$\operatorname{grad} f = \nabla_X X.$$

So it becomes obvious that zeros of X are critical points of f and that the critical set of f includes the orbits of X which are geodesics. The existence of an orbit α of X which is a geodesic imposes rank $(\nabla X)_\alpha \leqslant n - 1$.

The relation $df(X) = g(\nabla_X X, X)$ shows that if the quadratic form $Y_x \to g(\nabla_{Y_x} X, Y_x)$, $Y_x \in T_xM$ is positive definite for each $x \in M$, then critical points of f are zeros of X.

The definition of the Hessian of a real function and the definition of f imply.

Hess $f(Y, Y) = g(\nabla_Y(\nabla X)(Y), X) + g(\nabla_Y X, \nabla_Y X)$, $\forall Y \in \mathscr{X}(M)$. We will write down this Hessian in a form suitable to our arguments. Applying ∇_V to (1) and taking into account $\nabla g = 0$, we deduce

$$g(\nabla_V \nabla_Y X, Z) + g(\nabla_Y X, \nabla_V Z) - g(\nabla_V \nabla_Z X, Y) - \\ - g(\nabla_Z X, \nabla_V Y) = 0.$$

Since

$$\nabla_V \nabla_Y X = \nabla_V(\nabla_Y X) = \nabla_V(\nabla X)(Y) + \nabla X(\nabla_V Y) = \\ = \nabla_V(\nabla X)(Y) + \nabla_{\nabla_V Y} X,$$

$$g(\nabla_Y X, \nabla_V Z) = g(\nabla_{\nabla_V Z} X, Y)$$

we find

(2) $$g(\nabla_V(\nabla X)(Y), Z) - g(\nabla_V(\nabla X)(Z), Y) = 0, \ \forall V, Y, Z \in \mathscr{X}(M).$$

The Ricci identity

$$\nabla_V(\nabla X)(Z) - \nabla_Z(\nabla X)(V) = R(V, Z)(X)$$

may be written as

$$g(\nabla_V(\nabla X)(Z), Y) - g(\nabla_Z(\nabla X)(V), Y) = g(R(V, Z)(X), Y). \tag{3}$$

In the left side of (3) we add and substract $g(\nabla_V(\nabla X)(Y), Z)$,

$$g(\nabla_V(\nabla X)(Y), Z) - g(\nabla_V(\nabla X)(Y), Z) - g(\nabla_V(\nabla X)(Z), Y) -$$
$$- g(\nabla_Z(\nabla X)(V), Y) = g(R(V, Z)(X), Y).$$

Using (2) we obtain

$$g(\nabla_V(\nabla X)(Y), Z) = g(\nabla_Z(\nabla X)(V), Y) + R(Y, X, V, Z), \ \forall V, Y, Z \in \mathscr{X}(M).$$

All of these give

$$\text{Hess} f(Y, Y) = g(\nabla_X(\nabla X)(Y), Y) + R(X, Y, X, Y) + g(\nabla_Y X, \nabla_Y X),$$
$$\forall Y \in \mathscr{X}(M).$$

1. THEOREM. *Let X be an irrotational vector field on a Riemannian manifold (M, g) of dimension n and let f be the energy of X.*

(1) *If $x_0 \in M$ is a critical point of the energy f and rank $(\nabla X)_{x_0} = n$, then x_0 is a zero of X.*

(2) *Suppose rank $(\nabla X) \leqslant n - 2$.*

a) *If $x_0 \in M$ is a local minimum point of the energy f and the sectional curvature K_{x_0} of M at x_0 is strictly negative, then x_0 is a zero of X.*

b) *If $x_0 \in M$ is a local maximum point of the energy f and the sectional curvature K_{x_0} of M at x_0 is strictly positive, then X vanishes identically in a neighborhood of x_0.* ■

Proof. (1) From the expression of the gradient follows $\nabla_{X_{x_0}} X = 0$ and hence $X_{x_0} = 0$.

(2) Denote by x_0 a critical point of f, i.e. $\nabla_{X_{x_0}} X = 0$, which is not a zero of X, i.e. $X_{x_0} \neq 0$. As rank $(\nabla X) \leqslant n - 2$, there existe a vector field Y non-collinear with X such that $\nabla_Y X = 0$, i.e. $g(\nabla X, Y) = 0$. Fix Y by conditions $Y_{x_0} \in \text{Ker}\, (\nabla X)_{x_0}$ and $Y_{x_0} \perp X_{x_0}$, $Y_{x_0} \neq 0$. The relation $\nabla_Y X = 0$ implies $\nabla_X(\nabla X)(Y) + \nabla X(\nabla_X Y) = 0$ and hence $g(\nabla_X(\nabla X)(Y), Y) = -g(\nabla X(\nabla_X Y), Y) = -g(\nabla_{\nabla_X Y} X, Y) = 0$.

a) Using the fact that x_0 is a local minimum point we find $0 \leqslant \text{Hess} f(Y_{x_0}, Y_{x_0}) = K_{x_0} f(x_0) g(Y_{x_0}, Y_{x_0})$. It follows $f(x_0) \leqslant 0$ and hence $f(x_0) = 0$, contradicting $X_{x_0} \neq 0$. It remains $X_{x_0} = 0$.

b) As x_0 is a local maximum point we get $0 \geqslant \text{Hess} f(Y_{x_0}, Y_{x_0}) = K_{x_0} f(x_0) \times$

$\times g(Y_{x_0}, Y_{x_0})$. Consequently $f(x_0) \leqslant 0$ and hence $f(x_0) = 0$. Since $f(x_0) = 0$ is a local maximum, there exists necessarily a neighbourhood of x_0 on which f and hence X vanish identically. ■

2. COROLLARY. *Let (M, g) be a compact n-dimensional Riemannian manifold with sectional curvature strictly negative. If X is an irrotational vector field on (M, g) for which rank $(\nabla X) \leqslant n - 2$, then X possesses a zero.* ■

3. THEOREM. *Let X be an irrotational vector field on a complete Riemannian manifold (M, g). The energy f is convex if and only if*

$$g(\nabla_X(\nabla X)(Y), Y) + R(X, Y, X, Y) + g(\nabla_Y X, \nabla_Y X) \geqslant 0, \ \forall Y \in \mathscr{X}(M).$$

■

Let f be the energy of an irrotational vector field on (M, g). The trace of the Hess f is the Laplacian Δf. Therefore we obtain

$$(\Delta f)_x = \sum_{i=1}^{n} g(\nabla_{Y_i} X, \nabla_{Y_i} X) + S(X, X)_x + X_x(\operatorname{div} X),$$

where $Y_1, \ldots, Y_n$ is an orthonormal basis for $T_x M$ and S is the Ricci tensor field. We observe that $Q(Y, Y)_x = S(Y, Y)_x + Y_x(\operatorname{div} X)$, $Y_x \in T_x M$, defines an affine quadratic form $Q_x : T_x M \to R$ and note

$$\Gamma_x = \{ Y_x \in T_x M \mid Q(Y, Y)_x = 0\}, \ \Omega_x = \{ Y_x \in T_x M \mid Q(Y, Y)_x > 0\}.$$

4. THEOREM. *Let X be an irrotational vector field on the Riemannian manifold (M, g). If M is compact and $X_x \in \Gamma_x \cup \Omega_x$, $\forall x \in M$, then X is a parallel vector field and implicitly $X_x \in \Gamma_x$, $\forall x \in M$.* ■

Proof. Without loss of generality we may assume that M is orientable (if M is not orientable, we have only to consider an orientable double covering of M). $\Delta f \geqslant 0$ and the Green Theorem, $\int_M \Delta f \, dv = 0$, imply $\Delta f = 0$. It follows $g(\nabla_{Y_i} X, \nabla_{Y_i} X) = 0$, $i = 1, \ldots, n$ ($\Leftrightarrow \nabla X = 0$), $Q(X, X)_x = 0$. But the relation $Q(X, X)_x = S(X, X)_x + X_x(\operatorname{div} X) = 0$ is a consequence of $\nabla X = 0$, because in this case $\operatorname{div} X = 0$ and $S(X, X) = 0$. ■

Remark. If X is an irrotational vector field on (M, g), then cX, $c \in R$ is also irrotational. The expression of Q_x shows that $X_x \in \Gamma_x \cup \Omega_x$, $\forall x \in M$, implies $cX_x \in \Gamma_x \cup \Omega_x$, $\forall x \in M$. ■

5. COROLLARY. *If (M, g) is a compact Riemannian manifold and the Ricci tensor S_x is positive (negative) definite for each $x \in M$, then there do not exist nonzero irrotational vector fields on (M, g) with property $X_x \in \Gamma_x \cup \Omega_x$, $\forall x \in M$.* ■

Remark. In the hypothesis $\operatorname{div} X = 0$ (particular case of «$\operatorname{div} X$ is locally constant»), we find the results of Bochner [1] and Yano-Bochner [6]. ■

6. THEOREM. *Let X be an irrotational vector field on the Riemannian manifold (M, g). If each nonzero value X_x of X is in Ω_x and the energy f attains a local maximum at a point $x_0 \in M$, then X vanishes identically in a neighborhood of x_0.* ■

Proof. Suppose x_0 is a local maximum point (necessarily a critical point) of f and $X_{x_0} \neq 0$. The relation $\operatorname{Hess} f(Y_{x_0}, Y_{x_0}) \leqslant 0$, $\forall Y_{x_0} \in T_{x_0}M$ implies $(\Delta f)_{x_0} \leqslant 0$. On the other hand $\Delta f > 0$ in any point in which X is nonzero. It remains $X_{x_0} = 0$. As $f(x_0) = 0$ is a maximum, f and hence X must necessarily vanish identically on a neighborhood of x_0. ■

2. Let X be an arbitrary vector field on a Riemannian manifold, $f = \frac{1}{2} g(X, X)$ be the energy of X and $\alpha : I \to M$ be an orbit of X. The speed of variation of the energy f along orbits of X is given by the derivative $\nabla_X f = \mathrm{d}f(X) = g(\nabla_X X, X)$. If $\nabla_X X = 0$, then α is a geodesic. On the other hand, we observe that $g(\nabla_X X, X) \circ \alpha = (\|\nabla_X X\| \circ \alpha)(\|X\| \circ \alpha)$ if and only if along α we have $\nabla_X X = \frac{\mu}{2} X$, i.e. α is a geodesic reparametrized by $s = h(t)$, $t \in 1$, where

$$h(t) = a + b \int_{t_0}^{t} \exp\left(\frac{1}{2} \int_{t_0}^{r} \mu \circ \alpha(u) \mathrm{d}u\right) \mathrm{d}r, \quad a, b = \text{const.}$$

7. LEMMA. *Let X be a vector field on (M, g) and f be the energy of X. If α: $I \to M$ is an orbit of X, then*

$$f\circ\alpha(t)=\begin{cases} f\circ\alpha(t_0) & \text{if } \alpha \text{ is a geodesic}\\ f\circ\alpha(t_0)e^{\int_{t_0}^{t}\mu\circ\alpha(u)\mathrm{d}u} & \text{if } \alpha \text{ is a geodesic reparametrized by } s=h(t)\\ f\circ\alpha(t_0)+\int_{t_0}^{t} g(\nabla_X X, X)\circ\alpha(u)\,\mathrm{d}u & \textit{if otherwise} \end{cases}$$ ■

Proof. The result is a consequence of $\frac{\mathrm{d}}{\mathrm{d}t} f\circ\alpha = \nabla_X f\circ\alpha$. Really, if $\nabla_X X = 0$, then $\frac{\mathrm{d}}{\mathrm{d}t} f\circ\alpha = 0$; if α is a geodesic reparametrized by $s = h(t)$, $t\in I$, i.e. $\nabla_X X = \frac{\mu}{2} X$ equivalent to $g(\nabla_X X, X) = \|\nabla_X X\|\|X\|$, then $\frac{\mathrm{d}}{\mathrm{d}t} f\circ\alpha = (\mu\circ\alpha)(f\circ\alpha)$. ■

8. THEOREM. *Let X be a vector field on (M,g). If $g(\nabla_X X, X)$ does not vanish in any point, then the orbits of X cannot be closed (and hence nor periodic).* ■

Proof. Let $\alpha: I\to M$ be an orbit of X. Suppose there exist $t_1, t_2\in I$, $t_1<t_2$ so that $\alpha(t_1)=\alpha(t_2)$. If follows $f\circ\alpha(t_1) = f\circ\alpha(t_2)$. Taking Lemma 7 into account we find

$$\int_{t_1}^{t_2}\mu\circ\alpha(u)\mathrm{d}u = 0 \quad\text{or}\quad \int_{t_1}^{t_2} g(\nabla_X X, X)\circ\alpha(u)\mathrm{d}u = 0.$$

The mean value theorem on $[t_1, t_2]$ implies $g(\nabla_X X, X)\circ\alpha(u_0) = 0$, which contradicts the hypothesis. ■

9. THEOREM. *Let X be a vector field on (M,g) and $\alpha: I\to M$ an orbit of X. If α is a geodesic reparametrized by $s=h(t)$, $t\in I$ and $\mu\circ\alpha: I\to R$ is nondecreasing, then $f\circ\alpha: I\to R$ is either convex or concave. If α is neither a geodesic, nor a geodesic reparametrized by $s=h(t)$, $t\in I$ and $g(\nabla_X X, X)\circ\alpha: I\to R$ is nondecreasing, then $f\circ\alpha: I\to R$ is convex.* ■

Proof. We prove the second part. Lemma 7 and the hypotheses imply

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f\circ\alpha(t) = \frac{\mathrm{d}}{\mathrm{d}t} g(\nabla_X X, X)\circ\alpha(t) \geqslant 0, \quad \forall t\in I.$$ ■

10. THEOREM. *Let* $\dim M \geqslant 2$ *and let* X *be a vector field on* (M, g) *which admits an orbit* $\alpha : R \to M$ *with* $\alpha(R)$ *dense in* M.

(1) *If* α *is a geodesic reparametrized by* $s = h(t)$, $t \in I$, *then the integrals*

$$\int_{t_0}^{\infty} \mu \circ \alpha(u)\,\mathrm{d}u, \qquad \int_{-\infty}^{t_0} \mu \circ \alpha(u)\,\mathrm{d}u$$

are convergent and X *has no zeros on* M.

(2) *If* α *is neither a geodesic, nor a geodesic reparametrized by* $s = h(t)$, $t \in I$, *then the integrals*

$$\int_{t_0}^{\infty} g(\nabla_X X, X) \circ \alpha(u)\,\mathrm{d}u, \qquad \int_{-\infty}^{t_0} g(\nabla_X X, X) \circ \alpha(u)\,\mathrm{d}u$$

are convergent. ∎

Proof. The same arguments as in [4, Theorem 12].

(2) According to Lemma 7, the energy of X along the orbit α is

$$f \circ \alpha(t) = f \circ \alpha(t_0) + \int_{t_0}^{t} g(\nabla_X X, X) \circ \alpha(u)\,\mathrm{d}u.$$

Let $x \in M - \alpha(R)$. We may choose a sequence $\{U_k\}$ of neighborhoods of x such that $U_{k+1} \subset U_k$, which forms a basis at x. Any neighborhood of x includes some neighborhood U_k. Since each U_k contains a point of $\alpha(R)$, we may choose a sequence $\{s_k\}$ of real numbers such that $\alpha(s_k) \in U_k$. The sequence $\{s_k\}$ cannot be bounded. If $\{s_k\} \subset R$ where bounded, there would be a subsequence $\{s'_k\}$ with limit s and $x = \lim_{k \to \infty} \alpha(s'_k) = \alpha(s)$, i.e. $x \in \alpha(R)$, which contradicts the assumption $x \in M - \alpha(R)$. The sequence $\{\alpha(s_k)\}$ has limit x. Therefore

$$f(x) = \lim_{k \to \infty} f \circ \alpha(s_k) = \lim_{s_k \to \infty} f \circ \alpha(s_k) \text{ or } \lim_{s_k \to -\infty} f \circ \alpha(s_k),$$

i.e.

$$f(x) = f \circ \alpha(t_0) + \lim_{s_k \to \infty} \int_{t_0}^{s_k} g(\nabla_X X, X) \circ \alpha(u)\,\mathrm{d}u \text{ or}$$

$$f \circ \alpha(t_0) + \lim_{s_k \to -\infty} \int_{t_0}^{s_k} g(\nabla_X X, X) \circ \alpha(u)\,\mathrm{d}u.$$ ∎

11. THEOREM. *Let (M, g) be a compact Riemannian manifold, X be a vector field on (M, g), let f be the energy of X and α an arbitrary maximal orbit of X.*

(1) *Suppose α is a geodesic reparametrized be $s = h(t)$, $t \in I$. The integrals*

$$\int_{t_0}^{\infty} \mu \circ \alpha(u) \mathrm{d}u, \qquad \int_{-\infty}^{t_0} \mu \circ \alpha(u) \mathrm{d}u$$

are convergent and μ has a zero.

If the energy f does not vanish on M and zeros of μ are isolated, then $\alpha(R)$ joins two zeros of μ.

$$\lim_{t \to \infty} \int_{t_0}^{t} \mu \circ \alpha(u) \mathrm{d}u = -\infty \quad \text{or} \quad \lim_{t \to -\infty} \int_{t_0}^{t} \mu \circ \alpha(u) \mathrm{d}u = \infty$$

if and only if the point $x = \lim_{t \to \infty} \alpha(t)$ respectively $y = \lim_{t \to -\infty} \alpha(t)$ is a zero of f.

(2) *Suppose α is neither a geodesic, nor a geodesic reparametrized by $s = h(t)$, $t \in I$. The integrals*

$$\int_{t_0}^{\infty} g(\nabla_X X, X) \circ \alpha(u) \mathrm{d}u, \qquad \int_{-\infty}^{t_0} g(\nabla_X X, X) \circ \alpha(u) \mathrm{d}u$$

are convergent and $g(\nabla_X X, X)$ has a zero. If zeros of $g(\nabla_X X, X)$ are isolated, then $\alpha(R)$ joins two zeros of $g(\nabla_X X, X)$. ■

Proof. The same arguments as in [4, Theorem 13].

(2) Since M compact, the domain of α is R and the sequence $\{\alpha(k)\}$ possesses a convergent subsequence $\{\alpha(t_k)\}$ whose limit will be denoted by x. Taking Lemma 7 into account we find

$$f(x) = \lim_{k \to \infty} f \circ \alpha(t_k) = f \circ \alpha(t_0) + \lim_{k \to \infty} \int_{t_0}^{t_k} g(\nabla_X X, X) \circ \alpha(u) \mathrm{d}u.$$

If follows that the integral $\int_{t_0}^{\infty} g(\nabla_X X, X) \circ \alpha(u) \mathrm{d}u$ is convergent. The convergence of the integral and $\lim_{k \to \infty} g(\nabla_X X, X) \circ \alpha(t_k) = g(\nabla_X X, X)(x)$ imply $g(\nabla_X X, X)(x) = 0$.

The case of sequence $\{\alpha(-k)\}$ is analogous.

Under the hypothesis that zeros of $g(\nabla_X X, X)$ are isolated one proves that there exist zeros x and y such that $x = \lim_{t \to \infty} \alpha(t)$, $y = \lim_{t \to -\infty} \alpha(t)$.

It suffices to show that there exists an isolated zero of $g(\nabla_X X, X)$ such that $\lim_{t \to \infty} \alpha(t) = x$. If not so, there will be an $\epsilon > 0$ such that for each k, $d(\alpha(s_k), x) \geqslant \epsilon$ for some $s_k > t_k$. Since $d(\alpha(t_k), x) < \epsilon$ for k succifiently large, this says that $\alpha(t)$ enters and leaves the ball $\{ z \in M \mid d(z, x) < \epsilon \}$ repeatedly as $t \to \infty$. The distance $d(\alpha(t), x)$ must equal ϵ for some t between t_k and s_k; choose s_k so that in fact $d(\alpha(s_k), x) = \epsilon$. Since $\{ z \in M \mid d(z, x) = \epsilon \}$ is compact, the sequence $\{\alpha(s_k)\}$ has a subsequence converging to some point $x_1 \in M$ with $d(x_1, x) = \epsilon$. As $\lim_{k \to \infty} s_k = \infty$, the same argument that showed x was a zero of $g(\nabla_X X, X)$ also shows that x_1 is a zero of $g(\nabla_X X, X)$. Repeating this construction with ϵ replaced by $\frac{\epsilon}{m}$ leads to a zero $x_m \in M$ of $g(\nabla_X X, X)$ with $d(x_m, x) = \frac{\epsilon}{m}$, for each positive integer m. But this contradicts the fact that x is an isolated zero of $g(\nabla_X X, X)$. So, indeed, $\lim_{t \to \infty} \alpha(t) = x$.

Case of zero $y = \lim_{t \to -\infty} \alpha(t)$ is analogous. ∎

REFERENCES

[1] S. BOCHNER, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. 52 (1946), 776 - 797.

[2] Gr. TSAGAS, *On the singularities of harmonic 1-forms on a Riemannian manifold*, Kodai Math. Sem. Rep. 26, 4 (1975), 459 - 466.

[3] C. UDRIŞTE, *Extremum points of square lengths of some vector fileds*, National Colloquium of Geometry and Topology, Busteni, Romania, June 27 - 30, 1981; Bull. Math. Soc. Sci. Math. R.S. Romania (in press).

[4] C. UDRIŞTE, *Properties of conformal vector fields*, National Colloquium of Geometry and Topology, Cluj-Napoca, Romania, sept. 7 - 10, 1982; short communication, ICM, Warszawa, 1982.

[5] C. UDRIŞTE, *Properties of torse forming vector fields*, National Colloquium of Geometry and Topology, Piatra Neamt, Romaniaé June 16 - 19, 1983; Tensor, N.S., vol. 42 (1985), 137 - 144.

[6] K. YANO, S. BOCHNER, *Curvature and Betti Numbers*, Princeton Univ. Press, 1953.

Manuscript received: July 15, 1984.